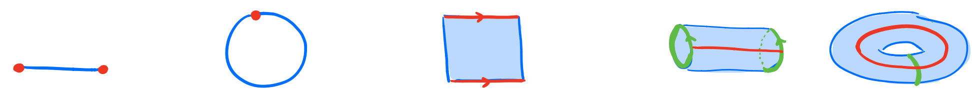
The Quotient Topology

We can get lots of interesting examples of topological spaces by "gluing" together simple spaces we already know

Ex:



$[0,1] \longrightarrow S^1$ $\qquad$ $[0,1]\times[0,1] \longrightarrow S^1\times[0,1] \longrightarrow S^1\times S^1$

We need to formalize this construction.

Def: Let X be a topological space and A a set. Let $f: X \longrightarrow A$ be a surjective function. The quotient topology on A is defined by $U \subseteq A$ is open $\Leftrightarrow f^{-1}(U)$ is open. (Exer: check this is a topology)

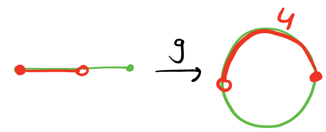
A map $f: X \to Y$ between topological spaces is a quotient map if f is surjective and $f^{-1}(U)$ is open $\Leftrightarrow U \subseteq Y$ is open.

Note that w/ the quotient topology on A, $f: X \to A$ is a quotient map. A is called a quotient space of X.

We can also construct A from X by introducing an equivalence relation $\sim$ on X and setting $A = X/\sim$. Then $f: X \longrightarrow A$ sends x to the equivalence class containing x.

Ex: We can think of S^1 as $[0,1]$ with 0 glued to 1.
i.e. the equivalence relation is just $0 \sim 1$, and the quotient map is $f(x) = (\cos 2\pi x, \sin 2\pi x)$.

Note that the map $g: [0,1) \to S^1$ defined as the restriction of f to $[0,1)$ is also surjective, but is not a quotient map:

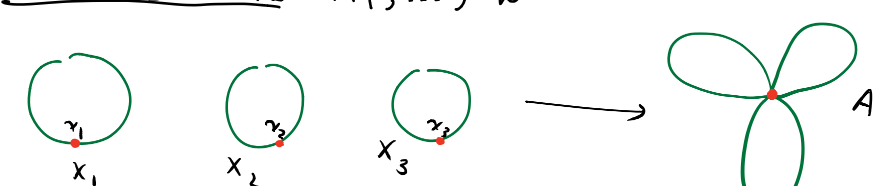


Let $U = \{(x,y) \mid y > 0\} \cup \{(0,0)\} \subseteq S^1$.

Then U is not open, but $g^{-1}(U) = [0, \frac{1}{2}) \subseteq [0,1]$ is open!
(whereas $f^{-1}(U) = [0, \frac{1}{2}) \cup \{1\}$ is not).

Ex: Let $(X_1, x_1), \dots, (X_n, x_n)$ be pointed topological spaces w/ X_i homeomorphic to S^1.

Then we get a quotient space A of $\sqcup X_i$ by the equivalence relation $x_i \sim x_j$ $\forall$ i,j. This is called the <u>wedge of the circles</u> $X_1, \dots, X_n$



A nice property of quotient maps is that if $h: X \to Y$ is a map that "respects the quotient structure" $X \to A$, we can uniquely define a map $A \to Y$. More precisely...

Thm: Let $p: X \longrightarrow Y$ be a quotient map. Let $f: X \longrightarrow Z$ be a continuous map to a topological space Z such that f is constant on each set $p^{-1}(\{y\})$ for y in z. (i.e. if $p(x_1) = p(x_2)$, then $f(x_1) = f(x_2)$). Then there is a continuous map $g: Y \to Z$ s.t. $g \circ p = f$. i.e.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "f"] & \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

commutes.

Pf: For each $y \in Y$, $f\left(p^{-1}(\{y\})\right)$ is a one-point set. Define $g(y)$ to be this point. Then if $x \in X$, $g(p(x)) = f(x)$ by construction. Thus, $g \circ p = f$.

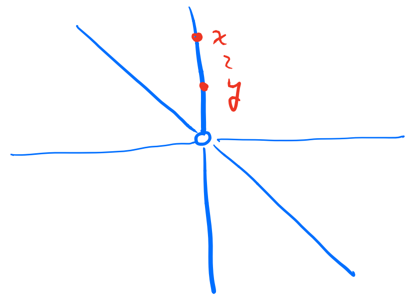
For continuity, let $U \subseteq Z$ be an open set. We want to show $g^{-1}(U)$ is open. $g^{-1}(U) \subseteq Y$ is open $\iff$ $p^{-1}\left(g^{-1}(U)\right) \subseteq X$ is open. But $p^{-1}\left(g^{-1}(U)\right) = (g \circ p)^{-1}(U) = f^{-1}(U)$, which is open since f is continuous. Thus $g^{-1}(U) \subseteq Y$ is open. $\square$

Note: If $f: X \to Y$ is surjective, continuous and open, then f is a quotient map. Similarly, if f is closed, then if $f^{-1}(U)$ is open, $X \setminus f^{-1}(U) = f^{-1}(Y \setminus U)$ is closed, so $Y \setminus U$ is closed $\Rightarrow U$ is open.

Thus f is also a quotient map in this case.

Example: Let $X = \mathbb{R}^n \setminus \{0\}$. We can put an equivalence relation on X as follows: $x \sim y \iff x = ay$ i.e. $x \sim y$ if and only if x and y lie on the same line through the origin.

You can check that this is an equivalence relation, and thus we get a corresponding quotient map $p: X \to X/\sim$.



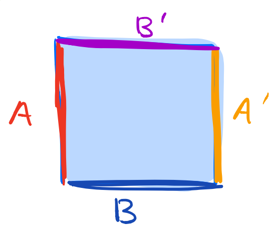
This is one way to construct <u>projective (n-1)-space</u>, defined

$$\mathbb{RP}^{n-1} := X/\sim, \quad \text{given the quotient topology.}$$

If Z is another topological space, then the continuous maps $f: X \to Z$ that give an induced map $\mathbb{RP}^{n-1} \to Z$ have the property that $f(\alpha x) = f(x)$ $\forall$ $\alpha \in \mathbb{R}\setminus\{0\}$, $x \in X$.

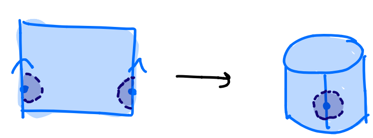
On the HW, we'll see that $\mathbb{RP}^1$ is homeomorphic to S^1. But we'll later see that $\mathbb{RP}^n$ is not homeomorphic to S^n for $n > 1$.

Example: Quotients of the unit square $X = [0,1]^2$



set
$A = \{0\} \times [0,1]$
$A' = \{1\} \times [0,1]$
$B = [0,1] \times \{0\}$
$B' = [0,1] \times \{1\}$

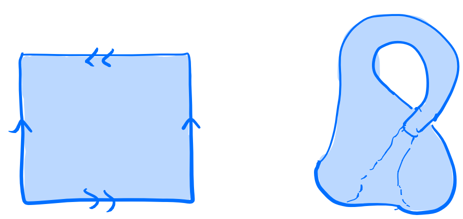
1.) If we glue A to A' via the equivalence relation $(0,t) \sim (1,t)$, we get a cylinder. A typical neighborhood of a point on the gluing line corresponds to two half moons along A and A' in X.



2.) If we glue A to A' via $(0,t) \sim (1, 1-t)$, we get a <u>Möbius band</u>.

3.) As we've seen, gluing A to A' and B to B' via $(0,t) \sim (1,t)$ and $(s,0) \sim (s,1)$ gives us the torus

4.) Gluing via $(0,t) \sim (1,t)$ and $(s,0) \sim (1-s, 1)$ gives us the <u>Klein Bottle</u>, which cannot be embedded in $\mathbb{R}^3$! We draw a picture where it overlaps itself, whereas an actual Klein bottle does not.



5.) Gluing $(0,t) \sim (1, 1-t)$ and $(s,0) \sim (1-s, 1)$ is a lot trickier to visualize! It turns out this space is homeomorphic to $\mathbb{RP}^2$!!

(not to hand in)

<u>Exercise</u>: Consider the quotient spaces $[0,1] \times [0,1] / \sim$ with the following equivalence relations.

a.) $(0,t) \sim (t,0)$

b.) $(0,t) \sim (t,0)$ and $(1,s) \sim (s,1)$

c.) (HARD!) $(t,0) \sim (0, 1-t)$

Can you visualize these spaces?